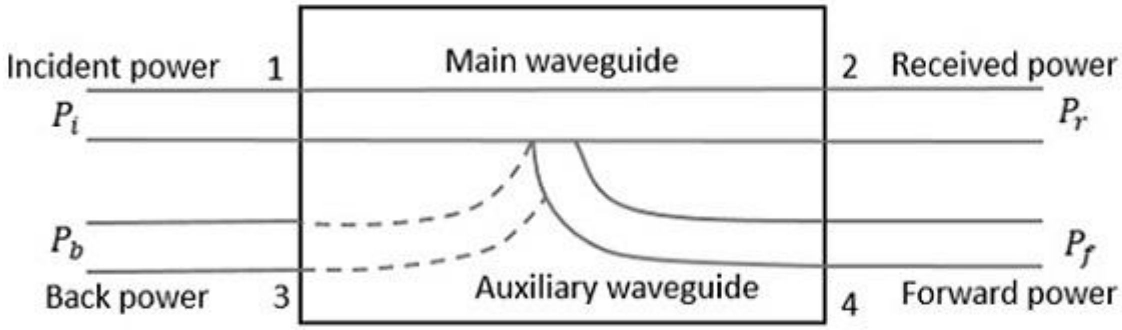
Directional Coupler S Matrix Derivation

Directional Coupler is a 4 port device which has primary and secondary waveguides. The primary wave guide is from port 1 to port 2 and secondary waveguide is from port 3 to port 4.



• Directional coupler is used to couple microwave power, which is unidirectional in most of the cases. The properties of a directional coupler are,

1. All the ports are matched.

2. When the power moves from port 1 to port 2, little portion of it would gets coupled to port 4 and not to port 3.

3. When the power moves from port 2 to port 1, little portion of it would gets coupled to port 3 and not to port 4.

4. The coupling factor of a directional coupler is the ratio of incident power to forward power.

Coupling factor = 10 $\log(P_i/P_f)$

• Directivity of the directional coupler is the ratio of forward power to back power.

Directivity = 10 $\log(P_f/P_b)$

• Isolation of a directional coupler is the ratio of incident power to back power.
I = 10 $\log(P_i/P_b)$

• Isolation = Coupling factor + Directivity.

• Two hole directional coupler is same as conventional directional coupler, but with two holes in common between primary and secondary waveguides.

The spacing between these two holes is given by,

$L = (2n+1)\lambda_g/4$

Where, n = an integer
λ_g = wavelength

• A fraction of energy entering into port 1 passes through holes and is radiated into port 2. The forward waves in port 4 are in the same phase and are added.

• The backward waves in port 3 are out of phase and are cancelled.

The general S matrix of a directional coupler is,

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} ------(1)$$

• Since all ports in a directional coupler are matched.

$S_{11} = S_{22} = S_{33} = S_{44} = 0$ ----------------- (2)

• Since there is no coupling between ports 1 & 3 and ports 2 & 4

$S_{13} = S_{31} = S_{24} = S_{42} = 0$ ------------- (3)

Apply equation (2) & (3) in (1)

$$S = \begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{12} & 0 & S_{23} & 0 \\ 0 & S_{23} & 0 & S_{34} \\ S_{14} & 0 & S_{34} & 0 \end{bmatrix}$$

By unitary property, [S][S]* = I

$$\begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{12} & 0 & S_{23} & 0 \\ 0 & S_{23} & 0 & S_{34} \\ S_{14} & 0 & S_{34} & 0 \end{bmatrix} \begin{bmatrix} 0 & S_{12}^* & 0 & S_{14}^* \\ S_{12}^* & 0 & S_{23}^* & 0 \\ 0 & S_{23}^* & 0 & S_{34}^* \\ S_{14}^* & 0 & S_{34}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1C_1 \Rightarrow |S_{12}|^2 + |S_{14}|^2 = 1$ ------------ (4)
$R_2C_2 \Rightarrow |S_{12}|^2 + |S_{23}|^2 = 1$ ------------ (5)
$R_3C_3 \Rightarrow |S_{23}|^2 + |S_{34}|^2 = 1$ ------------ (6)
$R_1C_3 \Rightarrow S_{12} S_{23}^* + S_{14} S_{34}^* = 0$ ----------- (7)

Comparing eq (4) and (5)

$|S_{12}|^2 + |S_{14}|^2 = |S_{12}|^2 + |S_{23}|^2$
$S_{14} = S_{23}$ --------------- (8)

Comparing eq (5) and (6)

$|S_{12}|^2 + |S_{23}|^2 = |S_{34}|^2 + |S_{23}|^2$
$S_{12} = S_{34}$ --------------- (9)

Let, S_{12} be real and positive,
ie, $S_{12} = S_{34} = p$ --------------- (10)

applying equation (10) in (7)

Therefore, $p\, S_{23}^* + S_{14}\, p = 0$
$p\, [S_{23}^* + S_{14}] = 0$
$p\, [S_{23}^* + S_{23}] = 0$
$S_{23}^* + S_{23} = 0$

To satisfy the above condition, S_{23} should be a complex value.

Let $S_{23} = jq$

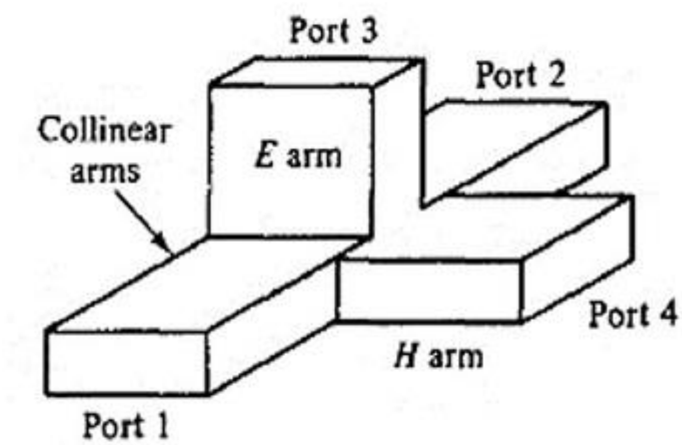
Therefore, the S matrix of directional coupler is,

$$S = \begin{bmatrix} 0 & p & 0 & jq \\ p & 0 & jq & 0 \\ 0 & jq & 0 & p \\ jq & 0 & p & 0 \end{bmatrix}$$

Magic Tee and Hybrid Ring S Matrix Derivation

Magic Tee or Hybrid Tee S Matrix Derivation

A combination of E plane Tee and M plane Tee is called Hybrid Tee or Magic Tee. It consists of four ports, if two waves of equal magnitude and same phase are fed into port 1 and port 2, the output will be subtractive and zero at port 3 and will be addictive at Port 4. A wave incident at Port 4 divides equally between port 1 and 2, and will not appear at port 3. A wave incident at Port 3 will produce an output of equal magnitude and opposite phase at ports 1 and 2. The magic tee is matched at ports 3 and 4.



Magic tee

The general matrix of the magic tee is given by,

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} ------(1)$$

From the property of Symmetry,
$S_{14} = S_{41}$,
$S_{13} = S_{31}$,
$S_{23} = S_{32}$ ---------------- (2)

Since port 3 acts as the E plane Tee,
$S_{13} = -S_{23}$ ------------------- (3)

Since port 4 acts as H plane Tee, $S_{14} = S_{24}$ ---------------- (4)

Considering the phase delay in the network,

$S_{34} = -S_{43} = 0$ and
$S_{12} = -S_{21} = 0$ ---------------- (5)

If port 3 and port 4 are matched,
$S_{33} = S_{44} = 0$ ---------------- (6)

Applying equation (2) to (6) in equation (1)

$$S = \begin{bmatrix} S_{11} & 0 & S_{13} & S_{14} \\ 0 & S_{22} & -S_{13} & S_{14} \\ S_{13} & -S_{13} & 0 & 0 \\ S_{14} & S_{14} & 0 & 0 \end{bmatrix} ------(7)$$

By unitary property, [S][S*] = I

$$\begin{bmatrix} S_{11} & 0 & S_{13} & S_{14} \\ 0 & S_{22} & -S_{13} & S_{14} \\ S_{13} & -S_{13} & 0 & 0 \\ S_{14} & S_{14} & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11}^* & 0 & S_{13}^* & S_{14}^* \\ 0 & S_{22}^* & -S_{13}^* & S_{14}^* \\ S_{13}^* & -S_{13}^* & 0 & 0 \\ S_{14}^* & S_{14}^* & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1C_1 \Rightarrow |S_{11}|^2+|S_{13}|^2+|S_{14}|^2 = 1$ -------------- (8)
$R_2C_2 \Rightarrow |S_{22}|^2+|S_{13}|^2+|S_{14}|^2 = 1$--------------- (9)
$R_3C_3 \Rightarrow 2|S_{13}|^2 = 1$
$S_{13} = 1/\sqrt{2}$ ------------- (10)
$R_4C_4 \Rightarrow 2|S_{14}|^2 = 1$
$S_{14} = 1/\sqrt{2}$ ---------------- (11)

Substitute, Equation (10) and (11) in equation (8)

$|S_{11}|^2+ (1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$
$|S_{11}|^2 = 1 – 1$
$\Rightarrow S_{11} = 0$

Equating equation (8) and (9)

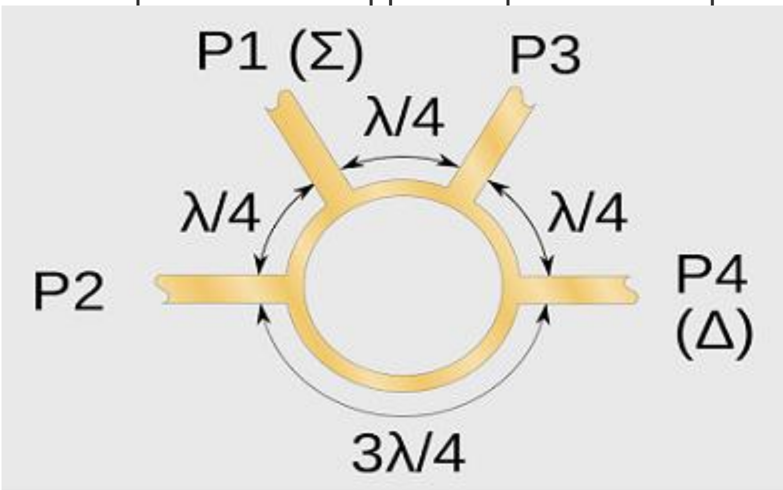
We get, $S_{11} = S_{22}$

Therefore the s matrix of magic tee is,

$$S = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}$$

Hybrid Ring S Matrix Derivation:

Hybrid ring circuits are also known as ‘Rat Race Coupler’. These junctions overcome the power limitations of magic tee. It is constructed by folding rectangular waveguides into circular waveguides. This junction has 4 ports with upper 3 ports separated by λ/4 and lower two ports separated by 3 λ/4. When a wave is fed into port 1, it will not appear at port 3 due to the phase shifts. Similarly wave fed onto port 2 will not appear at port 4 due to phase difference.



The general matrix of hybrid ring is,

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} ------(1)$$

If the ports 1,2,3 and 4 are matched then,
$S_{11} = S_{22} = S_{33} = S_{44} = 0$ -------------------- (2)

Considering the input – output conditions,
$S_{13} = S_{31} = 0$
$S_{24} = S_{42} = 0$
$S_{21} = -S_{41}$

Therefore, the general matrix can be written as,

$$S = \begin{bmatrix} 0 & S_{12} & 0 & -S_{12} \\ S_{12} & 0 & S_{23} & 0 \\ 0 & S_{23} & 0 & S_{34} \\ -S_{12} & 0 & S_{34} & 0 \end{bmatrix}$$

[S][S]* = I

$$\begin{bmatrix} 0 & S_{12} & 0 & -S_{12} \\ S_{12} & 0 & S_{23} & 0 \\ 0 & S_{23} & 0 & S_{34} \\ -S_{12} & 0 & S_{34} & 0 \end{bmatrix} \begin{bmatrix} 0 & S_{12}^* & 0 & -S_{12}^* \\ S_{12}^* & 0 & S_{23}^* & 0 \\ 0 & S_{23}^* & 0 & S_{34}^* \\ -S_{12}^* & 0 & S_{34}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_1C_1 =>

$|S_{12}|^2 + |S_{12}|^2 = 1$
$2|S_{12}|^2 = 1$
$S_{12} = 1/\sqrt{2}$

R_2C_2 =>

$|S_{12}|^2 + |S_{23}|^2 = 1$
$½ + |S_{23}|^2 = 1$
$S_{23} = 1/\sqrt{2}$

R_3C_3 =>

$|S_{23}|^2 + |S_{34}|^2 = 1$
$½ + |S_{34}|^2 = 1$
$S_{34} = 1/\sqrt{2}$

R_4C_4 =>

$|S_{12}|^2 + |S_{34}|^2 = 1$
$S_{12} = 1/\sqrt{2}$

Therefore, Matrix of Hybrid Rings is

$$S = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$$

Waveguide Tees in Microwave

Wave guide Tees in Microwave:

In microwave circuits, a wave guide with three independent ports is known as a ‘Tee’ junction. The characteristics of Tee junction are,

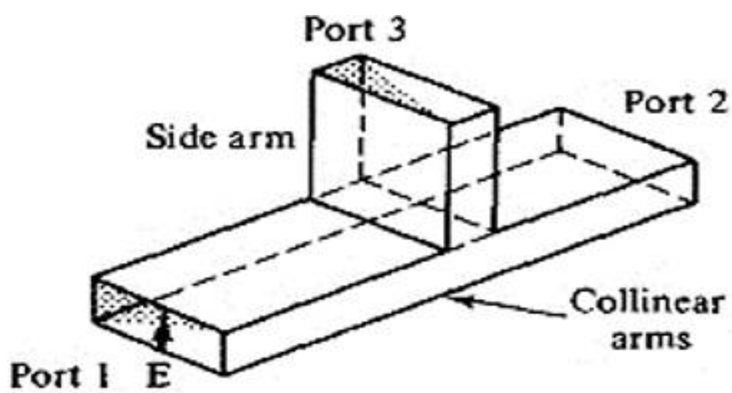
1. A short circuit can be positioned in one of the arms of 3 port junction in the manner that no power is moved through the other two arms.

2. If the junction is symmetric about one of its arms a short circuit can be placed on that arm. So that, no reflection occurs between other two arms.

3. It is impossible to have matched impedance on all the three arms.

E-Plane Tee in Microwave:

An E plane tee is a waveguide tee junction in which the axis of the side arm is parallel to the electric field. Two arms of the Tee junction are collinear arms. Signal entering at one port is divided among other two ports in such a way that the signals are out of phase with each other. The output of the E-plane will be the difference between input signals. The side arm of E-plane Tee is also known as difference arm.



E-plane tee

Since the E-Plane Tee is a 3 port network, the general ‘S’ matrix is represented as

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} ------(1)$$

The wave fed into port-3 appears at port 1 and port 2 with equal magnitude and opposite phase.

ie, $S_{13} = -S_{23}$ --------------------- (2)

If port 3 is matched, $S_{33} = 0$ --------------------- (3)

By the property of symmetry,
$S_{12} = S_{21}$
$S_{13} = S_{31}$
$S_{23} = S_{32}$ ---------------------- (4)

Applying equation (2), (3) & (4) in equation (1)

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & -S_{13} \\ S_{13} & -S_{13} & 0 \end{bmatrix} -------(5)$$

By unitary property '[S][S]* = I'

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & -S_{13} \\ S_{13} & -S_{13} & 0 \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* \\ S_{12}^* & S_{22}^* & -S_{13}^* \\ S_{13}^* & -S_{13}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_1C_1
=> $S_{11}S_{11}^* + S_{12}S_{12}^* + S_{13}S_{13}^* = 1$
=> $|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 = 1$ ------------ (6)

R_2C_2
=> $|S_{12}|^2 + |S_{22}|^2 + |S_{13}|^2 = 1$ ------------ (7)

R_3C_3
=> $|S_{13}|^2 + |S_{13}|^2 + 0 = 1$
=> $2|S_{13}|^2 = 1$
=> $S_{13} = 1/\sqrt{2}$ ------------ (8)

R_3C_1

=> $S_{13} S_{11}^* - S_{13}S_{12}^* = 0$
=> $S_{13} (S_{11}^* - S_{12}^*) = 0$
$S_{11}^* - S_{12}^* = 0$
$S_{11}^* = S_{12}^*$
ie, $S_{11} = S_{12}$ ------------------ (9)

Equating equations (6) and (7)

We get, $|S_{11}|^2 = |S_{22}|^2$

$S_{11} = S_{22}$ --------------------- (10)

Substitute equation (8) & (9) in equation (6)

$2|S_{11}|^2 + ½ = 1$

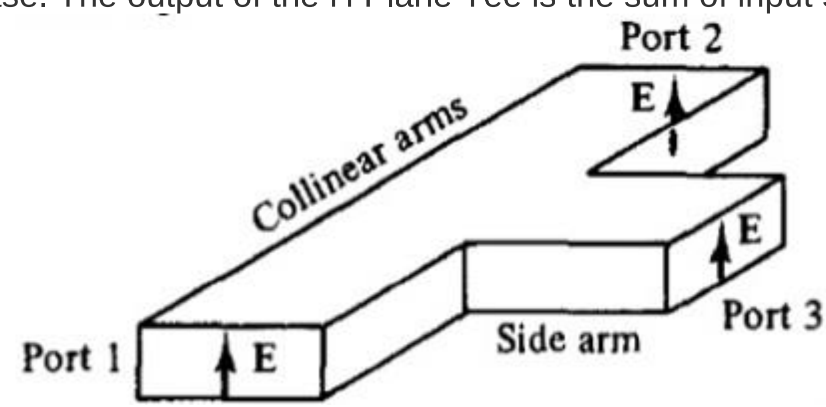
$S_{11} = ½$

Therefore, the scattering matrix of E plane Tee is,

$$S = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

H Plane Tee in Microwave:

In H Plane Tee, the side arm or H arm is parallel to the magnetic field. The signal fed to one of the ports will be divided between the other two ports and the signals will be in phase. The output of the H Plane Tee is the sum of input signals.



The general matrix is,

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} ------(1)$$

Since, the signals are in phase, $S_{13} = S_{23}$ ------------------- (2)

If Port 3 is matched, $S_{33} = 0$ -------------- (3)

By Symmetry,
$S_{12} = S_{21}$
$S_{13} = S_{31}$
$S_{23} = S_{32}$ ---------------------- (4)

Applying equation (2), (3) and (4) in (1).

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{13} \\ S_{13} & S_{13} & 0 \end{bmatrix} ------(5)$$

By unitary property, [S][S]* = I

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{13} \\ S_{13} & S_{13} & 0 \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* \\ S_{12}^* & S_{22}^* & S_{13}^* \\ S_{13}^* & S_{13}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_1C_1
=> $|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 = 1$ ------------ (6)

R_2C_2
=> $|S_{12}|^2 + |S_{22}|^2 + |S_{13}|^2 = 1$ ------------ (7)

R_3C_3
=> $|S_{13}|^2 + |S_{13}|^2 + 0 = 1$
=> $2|S_{13}|^2 = 1$
=> $S_{13} = 1/\sqrt{2}$ ------------ (8)

R_3C_1
=> $S_{13} S_{11}^* + S_{13}S_{12}^* = 0$
=> $S_{13} (S_{11}^* + S_{12}^*) = 0$
$S_{11}^* + S_{12}^* = 0$
$S_{11}^* = -S_{12}^*$
ie, $S_{11} = -S_{12}$ ------------------ (9)

Equating equation (6) and (7)

We get, $|S_{11}|^2 = |S_{22}|^2$

$S_{11} = S_{22}$ ------------------ (10)

Substitute eq (8) and (9) in eq (6)

We get,

$$S = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$